(7 Pages)

Reg. No. : ....

Code No. : 6846 **Sub. Code : PMAM 33**

M.Sc. (CBCS) DEGREE EXAMINATION, APRIL 2021.

Third Semester

Core — Mathematics

ADVANCED ALGEBRA — I

(For those who joined in July 2017 onwards)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If $\dim V = 6$ and $W = Hom(V,V)$ then $\dim Hom(W,F)$ is

   (a) 6 (b) 36×6

   (c) 36^2 (d) 36

2. If $(u,v) = i$ and $(u,w) = 2+i$ then $(u, iv + w)$ is

   (a) $2+3i$ (b) $3+i$

   (c) $1+i$ (d) $2+2i$

3. If $T \in A(V)$, then $\lambda \in F$ is called a characteristic root if

(a) $\lambda = T$ is regular

(b) for some $v \neq 0$ in V, $vT = v$

(c) $f(\lambda) = 0$ for some polynomial $f(x) \in F$

(d) $\lambda + T$ is singular

4. If $m(S) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $m(ST) = \begin{pmatrix} 3 & 6 \\ 5 & 12 \end{pmatrix}$ then $m(T)$ is

(a) $\begin{pmatrix} -1 & 0 \\ 3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$

(c) $\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} -1 & 0 \\ 3 & 4 \end{pmatrix}$

5. If $T \in A(V)$ is nilpotent, then 6 is called the index of nilpotence of T if

(a) $T^6 = 0$ but $T^5 \neq 0$ (b) $T^5 = 0$, $T^6 \neq 0$

(c) $T^6 = 0$ but $T^7 \neq 0$ (d) $T^6 = 0$

6. Which one of the following is not a Jordan block

(a) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$

(c) $\begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

7. If F is a field of characteristic 2 teritte matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ has trace

(a) 1 (b) 0

(c) 3 (d) 4

8. The secular equation of $\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ is

(a) $x^2 - x + 6$ (b) $x^2 - x$

(c) $x^2 - x - 6$ (d) $x^2 - 7$

9. Which one of the following is not true

(a) $(T^{\times})^* = T$ (b) $(S+T)^* = T^* + S^*$

(c) $(\lambda S)^* = \pi S^*$ (d) $(ST)^* = S^* T^*$

10. Which one of the following is not a characteristic root of an unitary transformation

(a) 1 (b) i

(c) $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$ (d) $1+i$

PART B — $(5 \times 5 = 25$ marks)

Answer ALL questions, choosing either (a) or (b).

Each answer should not exceed 250 words.

11. (a) If V finite-dimensional and $v \neq 0 \in V$, prove that there is an element $f \in V$ such that $f(v) \neq 0$.

Or

(b) If W is a subspace of an inner product space (finite dimensional) V, define $W^{\perp}$ and show that $(W^{\perp})^{\perp} = W$.

12. (a) If V is finite dimensional over F, prove that $T \in A(V)$ is regular if and only if T maps V onto V.

Or

(b) Let V be vector space of all polynomials over F of degree 3 or less and let D be its differentiation operator compute the matrix of D in the basis (i) $1, x, x^2, x^3$ (ii) $1, 1+x, 1+x^2, 1+x^3$.

13. (a) If $T \in A(V)$ is nilpotent, prove that $\alpha_0 + \alpha_1 T + \ldots + \alpha_m T^m$ where the $\alpha_i \in F$, is invertible if $\alpha_0 \neq 0$.

Or

(b) Suppose that $V = V_1 \oplus V_2$ where V_1 and V_2 are subspaces of V_1 invariant under T. Let T_1 and T_2 be the linear transformations induced by T on V_1 and V_2 respectively. If the minimal polynomial of T_i over F is $P_i(x), i = 1, 2$, prove that the minimal polynomial of T over F is the least common multiple of $p_1(x)$ and $p_2(x)$.

14. (a) For $A, B \in F_n$ and $\lambda \in F$, prove that $tr(\lambda A) = \lambda trA$ and $tr(AB) = tr(BA)$.

Or

(b) Prove that $\det(A) = \det(A')$.

15. (a) If $(vT, vT) = (v, v)$ for all $v \in V$, prove that T is unitary.

Or

(b) Let N be a normal transformation and suppose that λ and μ are two distinct characteristic roots of N. If v, w are in V and are such that $vN = \lambda v$, $wN = \mu w$, prove that $(v, w) = 0$.

PART C — ($5 \times 8 = 40$ marks)

Answer ALL questions, choosing either (a) or (b).

Each answer should not exceed 600 words.

16. (a) If V and W are of dimensions m and n, respectively, over F, prove that $Hom(V,W)$ is on dimension mn over F.

Or

(b) State and prove Schwary inequality.

17. (a) If V is finite dimensional over F, prove that $T \in A(V)$ is invertible if and only if the constant term of the minimal polynomial for T is not 0. And hence show that if $T \in A(V)$ is invertible then T^{-1} is a polynomial expression in T over F.

Or

(b) Let V be he vector space of polynomials of degree 3 or less over F and defined T on V by $(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3)T =$

$$\alpha_0 + \alpha_1(x+1) + \alpha_2(x+1)^2 + \alpha_3(x+1)^3.$$

Compute the matrix of D is the basis (i) $1, x, x^2, x^3$ (ii) $1, 1+x, 1+x^2, 1+x^3$ (iii) if the matrix in (1) is A and that in part (2) is B, find a matrix C so that $B = CAC^{-1}$.

18. (a) If $T \in A(V)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular.

Or

(b) Prove that two nilpotent linear transformations are similar if and only if they have the same invariants.

19. (a) For all $A, B \in F_n$, show that (i) $(A')' = A$ (ii) $(A+B)' = A' + B'$ and (iii) $(AB)' = B'A'$.

Or

(b) For $A, B \in f_n$, prove that $\det(AB) = (\det A)(\det B)$.

20. (a) Prove that the linear transformation T on V is unitary if and only if it takes an orthonormal basis of V into an orthonormal basis of V.

Or

(b) If $T \in A(V)$, prove that (i) $T^* \in A(V)$ (ii) $(T^*)^* = T$ and $(S+T)^* = S^* + T^*$.
